

# Topological spectrum of classical configurations

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**Abstract.** For any classical field configuration or mechanical system with a finite number of degrees of freedom we introduce the concept of topological spectrum. It is based upon the assumption that for any classical configuration there exists a principle fiber bundle that contains all the physical and geometric information of the configuration. The topological spectrum follows from the investigation of the corresponding topological invariants. Examples are given which illustrate the procedure and the significance of the topological spectrum as a discretization relationship among the parameters that determine the physical meaning of classical configurations.

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## INTRODUCTION

Canonical quantization is one of the keystones of modern physics, especially due to the fact that its application to gauge field theories has lead to the development of the standard model of elementary particles, whose predictions have been corroborated in the laboratory with spectacular accuracy. From a pure theoretical and mathematical point of view, nevertheless, canonical quantization presents severe problems. Among them one can mention the assumption of the existence of a Hilbert space that indeed exists only in very particular cases, the existence of operators that in the case of field configurations are not always well-defined, the problem of the classical limit, the non-uniqueness of the quantum evolution, the divergencies that require the application of different regularization and renormalization procedures, and many others [1, 2]. In the case of gravity, the situation is such that even if one would be able to solve all the technical problems of canonical quantization, one still would be confronted with the problem of time that confronts us with our conceptual understanding of space and time (see, for instance, [3] for a recent review).

In view of this situation, during the past 60 years many authors have been trying to formulate alternative quantization procedures (see, for instance, [2] for an introductory review). Unfortunately, none of these methods have reached the tremendous experimental success of canonical quantization. Recently, we started an alternative approach based upon the topological properties that can be extracted from a principal fiber bundle, associated to any classical physical configuration [5]. We called this method topological quantization. It has been used previously in the context of diverse monopole and instanton configurations [4]. We have shown that when applied to certain gravitational fields, topological quantization leads to a set of discretization conditions on the parameters that

determine the field. Moreover, a preliminary study [6] has shown that the method can be applied to mechanical systems with a finite number of degrees of freedom. In particular, it was possible to find discretization conditions that are equivalent to the spectrum following from the canonical quantization of the harmonic oscillator.

The aim of the present work is to begin a more strict formulation of the method of topological quantization. We introduce the concept of classical configuration, considering the geometric structures needed in topological quantization, and construct principal fiber bundles that can be associated to any given classical configuration. The concept of topological spectrum is explained by using topological invariants.

## CLASSICAL CONFIGURATIONS

Although the idea of classical configurations is very intuitive and no exact definition is usually necessary in physics, for the purposes of topological quantization we need a more mathematical approach. We introduce the concept of classical configuration as any classical (non quantum) physical system to which we can associate a unique geometric structure consisting of a differential manifold with a connection. Uniqueness should be understood here in a flexible manner as far as these two geometric objects are sufficient to distinguish classical configurations from each other. So, for instance, two differential manifolds which are related by an isomorphism with the same connection describe in our formalism the same classical configuration.

To be more specific let us consider examples from field theory. In Yang-Mills gauge theories, a classical configuration is a solution of the corresponding field equations. The differential manifold is the Minkowski spacetime and the connection  $A$  is a differential 1-form with values on the algebra  $\mathfrak{g}$  of the gauge group  $G$  which can be  $U(1)$ ,  $SU(2)$  or  $SU(3)$  for the known gauge interactions of nature. Any solution of the field equations can be represented, up to a gauge transformation, by means of a connection  $A$  which generates the gauge curvature  $F = dA$  in the Abelian case or  $F = dA + A \wedge A$  in the non-Abelian case. Although the physical information of the configuration is invariantly contained only in the gauge curvature, we will use the connection because, as we will see in the next section, its gauge freedom is an important component for the construction of the underlying geometric structure of topological quantization. A classical configuration in gauge field theories will then be denoted by  $(M_\eta, A)$ . Notice that the corresponding solution of the Yang-Mills equations does not necessarily need to be exact. Approximate solutions are also allowed, as far as they can be associated with gauge connections.

Gravitational fields are further examples of classical configurations. Let  $g$  denotes an exact or approximate solution of Einstein's equations in vacuum. According to general relativity, the Riemannian manifold  $M_g$  with metric  $g$  is the geometric object that contains all the relevant information about the gravitational field. In this case, a classical configuration will be denoted by  $(M_g, \omega)$  where  $\omega$  is the spin connection following from  $g$ . In fact, if we introduce a local orthonormal vierbein  $\theta^a$  by means of  $g = \eta_{ab} \theta^a \otimes \theta^b$ , where  $\theta^a = e^a_\mu dx^\mu$  and  $x^\mu$  are spacetime coordinates, then the spin connection is determined by  $d\theta^a = -\omega^a_b \wedge \theta^b$  and takes values on the algebra of the Lorentz group  $SO(1, 3)$ . The corresponding curvature 2-form  $\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$  is again the basic geometric object from which physical properties of the corresponding

gravitational field can be extracted. Notice that we have chosen a local description of the gravitational field in terms of the orthonormal differential frame  $\theta^a$ , instead of the usual tensorial approach with spacetime coordinates  $x^\mu$  for which we would have a classical configuration as the pair  $(M_g, \Gamma)$ , where  $\Gamma$  is the Levi-Civita connection. The advantage of the local approach in terms of differential forms is that we reduce the diffeomorphism invariance of the metric approach to the local invariance of the Lorentz group. The Lorentz invariance and the corresponding spin connection are clearly easier to handle from a geometric point of view. In contrast to gauge theories, in general relativity the metric  $g$  completely determines the Levi-Civita connection  $\Gamma$  (or the spin connection  $\omega$ ) so that for a classical configuration we only need to know  $g$ . Nevertheless, we use the notation  $(M_g, \omega)$  to emphasize the fact that it is possible to consider more general theories of gravity in which the connection is not compatible with the metric.

Mechanical systems with only a finite number of degrees of freedom can also be considered as classical configurations. Recall that in classical mechanics a system with  $k$  degrees of freedom is given by a Lagrangian of the form  $L = (1/2)g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - V(q)$  ( $\alpha, \beta = 1, \dots, k$ ). We limit ourselves to conservative systems in which the Hamiltonian is a constant of motion which coincides with the total energy  $E$  of the system. It turns out [6] that a way to differentiate mechanical systems from each other is through the Jacobi metric  $h = 2(E - V)g_{\alpha\beta}dq^\alpha dq^\beta$ , which is also used in Maupertuis' formulation of classical mechanics [7]. In fact, if one introduces Cartesian coordinates, the metric  $g_{\alpha\beta}$  becomes proportional to the Euclidean metric  $\delta_{\alpha\beta}$ , and the conformal factor  $(E - V)$  will contain in the potential all the information about the physical system. The properties of mechanical systems are invariant with respect to Galilean transformations. If we introduce a local orthonormal frame  $\theta^i$  such that  $h = \delta_{ij}\theta^i \otimes \theta^j$ , the invariance becomes reduced to that of the rotation group  $SO(k)$ . The differential 1-forms  $\theta^i$  generate a rotation connection  $\omega_j^i$ , as described above in the case of gravitational fields. Consequently, a mechanical system can be interpreted as a classical configuration with the pair  $(M_h, \omega)$ , where  $M_h$  is a  $k$ -dimensional conformally flat manifold with metric  $h$ . Notice that we do not need to know the solutions of the Euler-Lagrange equations in order to study mechanical systems as classical configurations. This is a property that could be of advantage, especially when investigating mechanical systems with potentials  $V(q)$  which do not allow an analytical integration of the motion equations.

The above examples show that the class of classical configurations includes all possible conservative mechanical systems, all vacuum gravitational fields, and all solutions of the Yang-Mills field equations. In the case of field configurations, one can now consider any combination of the gravitational field with different types of gauge matter to generate new classical configurations. One could say that any classical solution of the field equations for the four fields observed in nature, or an arbitrary combination of them, can be considered geometrically as a classical configuration.

It is worth mentioning that the scalar field in its standard Lagrangian description is not a classical configuration as defined above. There is no natural connection that could be associated to the scalar field, although it possesses the natural differential manifold structure of the Minkowski spacetime. In the context of topological quantization the scalar field requires a special treatment which will be presented elsewhere.

## TOPOLOGICAL SPECTRUM

According to our definition, a classical configuration is characterized by the pair  $(M, \omega)$  consisting of a differential manifold  $M$  and a connection 1-form  $\omega$ . Let us suppose that the physical content of  $(M, \omega)$  is invariant with respect to transformations of a group  $G$ . In the case of mechanical systems,  $G$  is clearly the Galilean group which reduces to the rotation group  $SO(k)$  when a local orthonormal vielbein is used in the description. In a similar manner, gravitational fields are in general locally invariant with respect to transformations of the Lorentz group  $SO(1, 3)$  acting on the differential 1-forms  $\theta^a$ . Gauge field configurations are invariant with respect to transformations of the corresponding gauge group, when acting on the connection  $A$ , and of the Lorentz group, when acting on the underlying Minkowski metric  $\eta$ . As we will see below only the gauge invariance is not trivial in this case.

We now use the triplet  $(M, \omega, G)$  to construct a principal fiber bundle  $P$  with a connection  $\tilde{\omega}$  in the following way. Let  $M$  be the base space of  $P$ . To each point of  $M$  we attach the elements of the group  $G$  as the typical fiber which is isomorphic to the structure group of  $P$ . Furthermore, let  $\sigma_i$  be the local section over an open subset  $U_i \subset M$  which follows in a standard manner [8] from the local trivialization of  $P$  over  $U_i \times G$ . The connection on  $P$  is introduced by means of the condition  $\sigma_i^* \tilde{\omega} = \omega_i$  where  $\omega_i$  is the connection  $\omega$  of the base space  $M$  evaluated on  $U_i$ , and  $\sigma_i^*$  is the pullback of  $\sigma_i$ . In [5] it was shown that these elements are sufficient to construct all the constituents of a principal bundle  $P$  with connection  $\tilde{\omega}$ . Moreover, for the triplet  $[M_g, \omega, SO(1, 3)]$ , which corresponds to a specific vacuum solution of Einstein's equations, one can prove the uniqueness of  $P$ . One could expect that an analogous proof could be formulated for gauge fields. In the case of mechanical systems a detailed proof is beyond the scope of the present work and will be presented elsewhere.

It is worth noticing that the old idea of additional dimensions to describe the physical behavior of fields is incorporated in a natural way in our construction. For instance, for gravitational fields we have that  $\dim(P) = \dim(M_g) + \dim[SO(1, 3)] = 10$ . A Yang-Mills field theory with gauge group  $SU(k)$  on the Minkowski spacetime  $M_\eta$  will be described on a principal fiber bundle of dimension  $4 + k^2 - 1$ . Classical configurations of the standard model of elementary particles with gauge group  $U(1) \times SU(2) \times SU(3)$  will be described on a 16-dimensional principal bundle. Mechanical systems with  $k$  degrees of freedom are characterized by a  $k$ -dimensional base space  $M_h$  with metric  $h_{ij}$  so that the dimension of the corresponding  $P$  is given by  $\dim M_h + \dim[SO(k)] = k + k(k-1)/2$ . Although those additional dimensions have very important physical consequences, in the sense that they are related to physical symmetries of the system, they cannot be observed directly on the spacetime which is the base space of  $P$ . In other words, our additional dimensions are important for the geometric construction of  $P$ , but they manifest themselves on the spacetime  $M$  only through conservation laws which can be associated to symmetries of the system.

Since to any classical configuration  $(M, G)$  we can associate a principal fiber bundle  $P$ , we can use the invariant properties of  $P$  to characterize each configuration. A characteristic class  $C(P)$  is a topological invariant of the bundle  $P$ ; in addition, the integral  $\int C(P)$  over the base manifold  $M$  (or over a compact cycle of  $M$ ) is also an invariant. The

remarkable result is that  $C(P)$  can always be normalized in such a way that  $\int C(P) = n$ , where  $n$  is an integer [9]. However, for this to be true, it is necessary that the integral could be computed, i.e.,  $C(P)$  must be a differential form. Fortunately, for the cases of interest in this work,  $C(P)$  can be expressed in terms of the curvature 2-form  $\Omega$ . The explicit form of  $C(P)$  depends on the structure group  $G$  of  $P$  (for more details see, for instance, [10]). If  $G = O(k)$ , one has the Pontrjagin class  $p(P)$ ; if  $G = SO(k)$ , one has the Pontrjagin class  $p(P)$  and the Euler class  $e(P)$  which is non-zero only when  $k$  is even; finally, for  $G = U(k)$  one obtains the Chern class  $c(P)$ . In all these cases, the integral  $\int C(P)$  can be calculated explicitly and the result is a function  $f(p_1, \dots, p_s)$  of the parameters that enter the metric of the base space  $M$  and, consequently, the curvature  $\Omega$ . So, as a result of the integration of the characteristic classes of  $P$  we obtain a relationship of the form

$$f(p_1, \dots, p_s) = n . \quad (1)$$

This is what we call the topological spectrum of the underlying classical configuration. It represents a discretization of the parameters  $p_1, p_2, \dots, p_s$  which determine the physical significance of the base space  $M$ . The following examples illustrate the result of applying this procedure to specific classical configurations.

Consider a mechanical system consisting of two harmonic oscillators of mass  $m$

$$L = \frac{1}{2}m[(\dot{q}^1)^2 + (\dot{q}^2)^2] - \frac{1}{2}[k_1(q^1)^2 + k_2(q^2)^2] . \quad (2)$$

The second oscillator is needed only to avoid the degeneracy the Jacobi metric for a single oscillator. When written in the local zweibein  $\theta^1 = \sqrt{2m(E-V)}dq^1$  and  $\theta^2 = \sqrt{2m(E-V)}dq^2$ , this classical configuration is invariant with respect to the group  $SO(2)$ . After the construction of the corresponding principal bundle  $P$  as described above, we find that the relevant characteristic class is the Euler class  $e(P) \propto \Omega_{12}\theta^1 \wedge \theta^2$ . Its integration in the limiting case of a single oscillator ( $k_2 = 0$ ,  $k_1 = k$ ) yields [6]

$$\frac{kq_0}{kq_0^2 - 2E} = n , \quad (3)$$

where  $q_0$  is a parameter related to the turning point of the oscillator. This is the topological spectrum of the harmonic oscillator. The constant  $q_0$  can be chosen such that one obtains the canonical spectrum from the topological one.

Consider the gravitational field of the Kerr-Newman black hole which is completely determined by the mass  $m$ , specific angular momentum  $a$ , and charge  $e$ . It is a solution of the Einstein-Maxwell equations. The base space of  $P$  corresponds to the spacetime  $M_g$  for this black hole. For the structure group we have two immediate possibilities: either  $G = SO(1, 3)$  or  $G = U(1)$ . For the sake of simplicity, let us consider the 5-dimensional principal bundle  $P$  with structure group  $U(1)$ . The connection  $A$  is a  $u(1)$ -connection from which the gauge curvature  $F$  can be computed. The Chern class  $c(P)$  is in this case the relevant characteristic class whose integration over the base space results in the topological spectrum [5]

$$\frac{2e^3\sqrt{m^2 - e^2 - a^2}}{r_0(e^4 + 4m^2a^2)} = n , \quad (4)$$

where  $r_0$  is an integration constant. In the case of vanishing angular momentum,  $a = 0$ , this topological spectrum can be rewritten in terms of the horizon area  $\mathcal{A}$  of the Reissner-Nordström black hole as

$$\mathcal{A} = 4\pi e^2 \mathcal{A}_0 \left[ \frac{n}{2} + \sqrt{1 + \frac{n^2}{4}} \right]^2, \quad (5)$$

where  $\mathcal{A}_0$  is a constant. We see that in this case the topological spectrum corresponds to a discretization of the horizon area. Further examples of gravitational classical configurations are given in [5].

In the case of the principal bundle for classical gauge configurations we also have two different possibilities for the structure group: the Lorentz group which represents the invariance of the background Minkowski spacetime, and the gauge group which corresponds to the gauge invariance of the field. Since the spin connection of the background metric is flat, the corresponding characteristic class vanishes identically and no topological spectrum is obtained. However, if we take the gauge group as the structure group of  $P$  a topological spectrum emerges. Probably, the simplest example of a topological spectrum for a gauge theory is the one which follows from analyzing the field of an electric charge [9]. The resulting principal bundle  $P$  is a  $U(1)$ -bundle whose Chern number implies the topological quantization of the electric charge.

## CONCLUSIONS

In the present work, we presented a geometric definition of classical configurations which leads to the natural introduction of a principal fiber bundle with a connection. It can be applied to solutions of differential field equations, such as the gauge fields and the gravitational field, and mechanical systems with a finite number of degrees of freedom. We showed that the study of the corresponding characteristic classes of the principal bundle leads to a topological spectrum, represented by a discretization relationship among the parameters which determine the physical significance of the underlying classical configuration.

## REFERENCES

1. R. M. Wald, *Quantum field theory in curved space-time and black hole thermodynamics*, University Press, Chicago, USA, 1994.
2. S. Carlip, *Rept. Prog. Phys.* **64**, 885 (2001).
3. A. Macias and H. Quevedo, "Time paradox in quantum gravity," in *Quantum gravity*, edited by B. Fauser et al., Birkhäuser Verlag, Basel, Switzerland, 2007.
4. T. Frankel, *The Geometry of Physics*, Cambridge University Press, Cambridge, UK, 1997.
5. L. Patiño and H. Quevedo, *J. Math. Phys.* **46**, 22502 (2005).
6. F. Nettel and H. Quevedo, *Topological quantization of the oscillator*, arXiv:math-ph/0612038.
7. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Verlag, New York, 1989.
8. G. L. Naber, *Topology, Geometry, and Gauge Fields*, Springer Verlag, New York, 1997.
9. Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics*, Elsevier Science Publishers, Amsterdam, 1982.
10. C. Nash and S. Sen, *Topology and geometry for physicists*, Academic Press, London, UK, 1983.